The multi-path Traveling Salesman Problem with dependent random cost oscillations

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1 Introduction

Reducing air pollution is nowadays one of the main goals of several institutions and governments of industrial countries. Developing methods able to schedule companies logistic operations and public services in a efficient way is coherent with such objective as it would imply less production of pollutants. Several logistic problems can be assimilated to the traveling salesman problem (TSP). Nevertheless, in real applications the stochastic environment deeply influences the performance of the solution. In [1] it has been proven that it is important to consider the stochastic nature of the distribution network in order to obtain affordable solutions. Nevertheless, since the complexity of the stochastic problem grows very fast, deterministic approximations have been developed. All these approximations assume that cost oscillations are independent and identically distributed. In this paper, we relax such hypothesis by assuming that cost oscillations are still identically distributed but just asymptotically independent. This assumption addresses the traffic congestion effects. In particular, we consider the multi-path traveling salesman problem with dependent random cost oscillations ($mpTSP_{do}$), where the cost oscillations are stochastic with unknown distribution and between any pair of locations there are usually several alternative paths. We propose a method to find a deterministic approximation solution of the $mpTSP_{do}$ and we evaluate its quality and efficiency.

2 The stochastic problem

Let us consider a graph network characterized by a set of locations N, a set of scenarios S and a set of all possible paths $P = \{P_{ij}\}$ connecting location $i \in N$ to location $j \in N$. Further, we consider c_{ij}^p the unit deterministic cost associated to the path $p \in P_{ij}$, Θ_{ij}^{ps} the random oscillation of the deterministic cost c_{ij}^p under scenario $s \in S$ and $C_{ij}^{ps}(\Theta) = c_{ij}^p + \Theta_{ij}^{ps}$ the total unit cost required for travelling from the location i to the location j using path $p \in P_{ij}$ under scenario s. Finally, let us consider variable x_{ij}^p equals 1 if path $p \in P_{ij}$ is selected and 0 otherwise and variable y_{ij} equals 1 if location j is visited directly after location i and 0 otherwise. The model of the multi-path traveling salesman problem with dependent travel cost oscillations problem is

$$\min_{x,y} \ \mathbf{E}_\Theta \left[\sum_{i \in N} \sum_{j \in N} y_{ij} \sum_{p \in P_{ij}} \sum_{s \in S} x_{ij}^p C_{ij}^{ps}(\Theta) \right] \tag{1}$$

subject to

$$\sum_{j \in N, j \neq i} y_{ij} = 1 \quad \forall i \in N \tag{2}$$

$$\sum_{i \in N, i \neq j} y_{ij} = 1 \quad \forall j \in N \tag{3}$$

$$\sum_{i \in U} \sum_{j \in U} y_{ij} \geq 1 \quad \forall \ U \subset N \tag{4}$$

$$\sum_{p \in P_{ij}} x_{ij}^p = y_{ij} \quad \forall i \in N \ \ j \in N \tag{5}$$

$$x_{ij}^p \in \{0,1\} \ \ \forall p \in P_{ij} \ \ \forall i \in N \ \ \forall j \in N \tag{6}$$

$$y_{ij} \in \{0,1\} \ \ \forall i \in N \ \ \forall j \in N \tag{7}$$

The objective function (1) minimizes the expected total cost, constraints (2) and (3) ensure that each location is visited once, while constraints (4) prevent the formation of subtours. Finally, constraints (5) link variables x_{ij}^p to y_{ij} and (6) and (7) are the integrality constraints.

3 A deterministic approximation of the stochastic problem

Following the approach of [1] we consider that the cost required for traveling from location i to location j is

$$C_{ij}(\Theta) = \min_{p \in P_{ij}} \left(c_{ij}^p + \min_{s \in S} \ \Theta_{ij}^{ps} \right) \tag{8}$$

From this consideration, the variables x_{ij}^p assume value 1 iff p is the cheapest path. Moreover, from Eq. (8) and from the linearity of the expected value the objective value function

(1) becomes

$$\min_{y} \sum_{i \in N} \sum_{j \in N} \mathbf{E}_\Theta \left[C_{ij}(\Theta) \right] y_{ij} \tag{9}$$

Unfortunately, the distributions of the Θ_{ij}^{ps} are unknown and thus the expected value in (9) is not solvable. In [1] the authors assume that the random cost oscillations Θ_{ij}^{ps} are independent and identically distributed random variables with unknown probability distribution. Under a mild assumption on that unknown probability distribution, they give a deterministic approximation of the stochastic problem.

In this paper we want to relax the independence assumption of the random cost oscillations. We assume that these oscillations are not anymore independent but just asymptotically independent. This very mild assumption addresses the traffic congestion effects where these oscillations cannot be considered independent any more.

Definition 3.1 *Let X_1 and X_2 be random variables. They are asymptotically independent if*

$$\lim_{r \to -\infty} \left(P(X_1 < r | X_2 < r) - P(X_1 < r) \right) = 0 \tag{10}$$

It is worth noting that by subtracting from all random cost oscillations Θ_{ij}^{ps} a constant $\alpha_{|S|}$ and by dividing the resulting variables by any other constant $\gamma_{|S|} > 0$ the solution of the problem does not change. Hence, the cost $C_{ij}(\Theta)$ becomes

$$C_{ij}(\Theta) = \min_{p \in P_{ij}} \left(c_{ij}^p + \min_{s \in S} \left(\frac{\Theta_{ij}^{ps} - \alpha_{|S|}}{\gamma_{|S|}} \right) \right) \tag{11}$$

Let $F_{ij}^p = P(\Theta_{ij}^{ps} > x)$ be the unknown survival function of the random oscillations Θ_{ij}^{ps}, We proved in [2] the following theorem

Theorem 3.1 *Assuming that the random cost oscillations $\Theta_{ij}^{p_1 s}$ and $\Theta_{ij}^{p_2 s}$ are asymptotic independent $\forall\ p_1,\ p_2 \in P_{ij}$ under each scenario s and that we choose $\alpha_{|S|}$ and $\gamma_{|S|}$ such that*

$$\lim_{|S| \to \infty} \left(F_{ij}^p \left(\gamma_{|S|} x - \alpha_{|S|} \right) \right)^{|S|} = \exp\left(-e^{\beta x} \right) \quad \textit{for some real number} \ \ \beta > 0, \tag{12}$$

then

$$\lim_{|S| \to \infty} P(C_{ij}(\Theta) > x) = e^{-A_{ij} e^{\beta x}}, \quad \textit{where} \ \ A_{ij} = \sum_{p \in P_{ij}} e^{-\beta c_{ij}^p} \quad \forall i \in N \ \ \forall j \in N \tag{13}$$

It is worth noting that (12) is a mild assumption as, for suitable constants $\alpha_{|S|}$ and $\gamma_{|S|}$, it holds for several distributions. If $|S|$ is large enough, the limit obtained in (13) can be used as the survival function of the costs $C_{ij}(\Theta)$. Thus, after some manipulations, the expected value in (9) becomes

$$\mathbf{E}_\Theta \left[C_{ij}(\Theta) \right] \approx \int_{-\infty}^{+\infty} x e^{-A_{ij} e^x} A_{ij} e^x dx = -\frac{1}{\beta} (\ln(A_{ij}) + \gamma) \tag{14}$$

where $\gamma = -\int_0^\infty \log(t)e^{-t}dt \approx 0.5772$ is the Euler constant.

Using (14) and disregarding the constants the following deterministic approximation of the stochastic problem is finally obtained

$$\min_y \sum_{i\in} \sum_{j\in N} -\frac{1}{\beta} y_{ij} \ln A_{ij}, \quad \text{s.t. } (2),\ (3),\ (4),\ (7) \tag{15}$$

4 Computational results and conclusions

We compute some numerical experiments in other to evaluate the effectiveness of our deterministic approximation. In particular, we compare the solution of our deterministic approximation with the one of the Perfect Information case, computed by means of a Monte Carlo simulation performed on the stochastic problem. The percentage gap is defined as the relative percentage difference between the optimum obtained with the deterministic approximation and the one provided by the Monte Carlo simulation. In the experiments we used 20 locations and consider the Θ to be distributed according to a multivariate normal. For the sake of space we present here just a few results in Table 1. For the total number of scenarios $|S| = 100$, the percentage gap turns out to be 3.7%, but it significantly decreases as the number of paths between pairs of locations increases. It is important to note that on average the deterministic approximation needs only 0.1 seconds to be computed, while the stochastic approach needs 36 seconds.

Table 1: Number of paths and percentage gap obtained by using $|S| = 100$

number of paths	percentage gap
1	3.7
3	2.25
5	0.76

References

[1] Roberto Tadei, Guido Perboli, Francesca Perfetti (2017) *The multi-path Traveling Salesman Problem with stochastic travel costs.* EURO Journal on Transportation and Logistics, vol. 6 n. 1, 3-23.

[2] Edoardo Fadda, Lohic Fotio Tiotsop, Guido Perboli, Roberto Tadei (2017) *The stochastic Traveling Salesman Problem with dependent random cost oscillations.* DAUIN internal report.